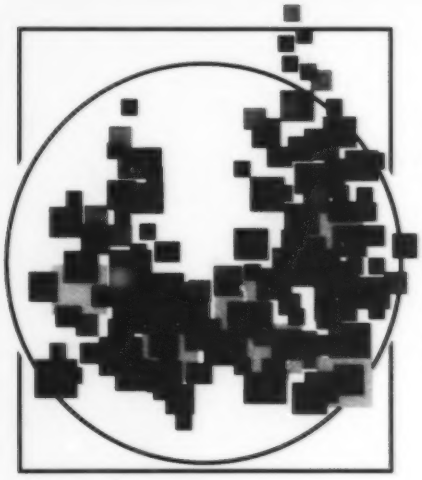

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On the Structure of the Matrix Riccati Equation and the Associated LQR

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ABSTRACT

The problem of selecting weighting matrices for an optimal linear quadratic regulator (LQR) such that the closed loop eigenspectrum are assigned to desired locations is addressed. The approach presented is a recursive one where at each stage of the recursive procedure the dynamic system is aggregated to a second or first order system depending on whether a complex conjugate pair or a real pole is being placed. The approach is capable of assigning all or a subset of the poles without altering the remaining open loop poles. In addition the optimal controller is capable of placing a complex conjugate pair at a new complex conjugate location, or at two distinct real locations. The algorithm is computationally attractive since low order matrix calculations are required throughout.

1.0 - INTRODUCTION

It is well known that the optimal linear quadratic regulator (LQR) theory is one of the most powerful approaches to multivariable control system design for variety of applications. Its dual problem also is widely used in optimal filtering. The theory is elegant and interesting since necessary and sufficient conditions can be stated throughout its development. Among several attractive features of the LQR problem are the fact that under certain stabilisibility and detectability conditions, the resulting closed loop system is asymptotically stable, and has certain desirable inherent properties. That is, the resulting system will possess infinite gain and at least 60 degree phase margin, which implies that it has good sensitivity and robustness properties.

The one shortcoming of the theory has been the fact that it does not provide any insight into the resulting closed loop system (i.e. such

issues as the transient response behavior, and the location of closed loop system poles) beforehand. Also there is not a clear cut procedure on how to select the weighting matrices in the quadratic cost functional. This is an important issue, since these weights have direct bearing on the location of the closed loop poles, and the response of the system. Traditionally the weighting matrices in the quadratic cost functional are selected based on some trial and error procedure, so that an acceptable set of closed loop system eigenspectrum, or desired response is achieved.

The problem of selecting weights for an optimal linear regulator, and its relationship to the closed loop system eigenspectrum has been the subject of many studies in recent years (see Johnson, and Grimble (1987), and references cited). Basically there have been two approaches to the problem. The first class of these are those techniques that assign the closed loop poles of the dynamic system to a region in the left hand plane, originated by the desire to design a regulator with a prescribed degree of exponential stability, and was proposed by Anderson and Moore (1971). This was done by extending the LQR theory to place its closed loop eigenspectrum to the left of the vertical line $-\epsilon$, ($\epsilon>0$). Further works are those of Kawasaki and Shimemura (1983, 1988) that uses a recursive procedure to place the poles of the optimal regulator in an open hyperbola, while in shieh et.al. (1986) they are placed within a vertical strip in the left hand complex plane. A transformation approach is used by Furuta and Kim (1987, 1988), and Wittenmark et.al. (1987) which places the entire closed loop eigenspectrum of the closed loop system in a specified disk in the left hand plane.

The Approach taken in this study falls under the second category of approaches, and that is those studies which look for assigning the eigenspectrum of the closed loop system to exact predefined locations,

while minimizing the performance index. In particular, here a relationship between the weighting matrices in the quadratic cost functional, and the closed loop system's eigenspectrum is of interest. The work of Solheim (1972), Graupe (1972), Harvey and Stein (1978), Grimble (1981), Amin (1985), and Medanic et. al. (1987) are examples of such approaches.

In this study a recursive procedure for determining the elements of the matrix of weights on the state variables (Q matrix), for a desired set of eigenvalues of the optimal regulator is presented. This research study is basically an extension of the work by Amin (1985), and Medanic et.al. (1986). Specifically the algorithm presented is superior to these works in the sense that in the case of the complex conjugate pair of poles it is possible to achieve placement of both real and imaginary parts of the poles, whereas in the scheme of Amin (1985), and Medanic et.al. (1986) it is not possible to change the natural frequencies of oscillation. In the approach presented certain frequencies of oscillation can be achieved, and in addition it would be possible to transform a pair of complex conjugate poles into two real poles placed at certain desired positions. Another advantage of this scheme is that at each stage of the recursive procedure it is necessary to work with low order matrices (second or fourth order) regardless of the dimension of system. It should be pointed out that for the sake of simplicity here it is assumed that the system under consideration has no repeated roots.

2.0 - STATEMENT OF THE PROBLEM

Consider the following linear dynamical system described by the state space formulation

$$\dot{\underline{X}} = \underline{A}\,\underline{X} + \underline{B}\,\underline{U} \qquad \text{with } \underline{X}(0) = \underline{X}_o \qquad (1)$$

where $\underline{X} \in \mathbb{R}^n$, and $\underline{U} \in \mathbb{R}^m$ are the state and the control respectively, and matrices $\underline{A}$ and $\underline{B}$ are of appropriate dimensions. An optimal control law is desired such that the following performance index is minimized subject to (1)

$$J = \frac{1}{2} \int_0^\infty \left(\|\underline{X}\|^2_{\underline{Q}} + \|\underline{U}\|^2_{\underline{R}} \right) dt \tag{2}$$

where $\underline{Q}$ and $\underline{R}$ are weighting matrices of appropriate dimension and $\underline{R}$ needs to satisfy the usual definiteness assumption, however $\underline{Q}$ as will be seen later need not satisfy the customary positive semidefiniteness assumption. It is further assumed that the system is completely controllable and the controls are not bounded. The solution to the above problem can be obtained by using the optimality conditions, as

$$\underline{U} = -\underline{R}^{-1} \underline{B}^T \underline{\psi} = -\underline{R}^{-1} \underline{B}^T \underline{P} \underline{X} \tag{3}$$

$$\dot{\underline{\psi}} = -\underline{Q} \underline{X} - \underline{A}^T \underline{\psi} \tag{4}$$

where $\underline{\psi} \in \mathbb{R}^n$ is the co-state vector, and $\underline{P}$ is the symmetric positive semidefinite solution of the algebraic matrix Riccati equation (AMRE), given by

$$\underline{P} \underline{A} + \underline{A}^T \underline{P} - \underline{P} \underline{B} \underline{R}^{-1} \underline{B}^T \underline{P} + \underline{Q} = 0 \tag{5}$$

The following gives further insight into the structure of the AMRE and also provides an alternative solution to it.

THEOREM 1- $\underline{P} \geq 0$ exist if and only if : 1) the $\mathrm{Re}(\lambda_i) \neq 0$ for all eigenvalues λ_i of the 2n x 2n Hamiltonian matrix H formed by augmenting (1) and (4) as

$$\dot{\underline{\xi}} = \underline{H} \underline{\xi}$$

$$\xi = \begin{pmatrix} X \\ \psi \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} A & -B R^{-1} B^T \\ -Q & -A^T \end{pmatrix} \tag{6-a}$$

and 2) (A,B) is stabilisable.

Proof: For a proof see Potter (1966), Martensson (1971), and Kucera (1972).■

The eigenstructure of H can be determined by

$$\begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \Pi^-(H) & 0 \\ 0 & -\Pi^-(H) \end{pmatrix}$$

where $\Pi(.)$ refers to a diagonal matrix having the eigenvalues of the matrix argument as its diagonal elements, and $\Pi^-(H) \epsilon R^{n \times n}$ is a diagonal matrix containing the stable eigenvalues of H . The solution to P is given by

$$P = b\, a^{-1} \tag{6-b}$$

where the n x n matrices a, and b are given by $a = [\, a_1, a_2, \ldots, a_n]$ $b = [\, b_1, b_2, \ldots, b_n]$, and $v_i = (\, a_i \quad b_i)^T$ is an eigenvector (or generalized eigenvector) associated with stable λ_i's of H.

It is well known that the resulting closed loop optimal system is asymptotically stable and is given by

$$\dot{X} = A_c\, X \quad \text{where } A_c = A - B\, R^{-1} B^T P \tag{7}$$

Let $\Lambda(A) = \{\, \lambda_i^o,\ i = 1,2,\ldots,n\}$, and $\Lambda(A_c) = \{\, \lambda_i^c,\ i = 1,2,\ldots,n\}$ be the set of open and closed loop eigenvalues of the system respectively.

Lemma 1- It is a property of (4) that its solution P satisfies

$$\det(sI - A + B\, R^{-1} B^T P\,)\, \det(sI + A - P\, B\, R^{-1} B^T) = \det(sI - H\,)$$

that is the elements of $\Lambda(A_c)$ are the stable eigenvalues of the symplectic

matrix $\underline{H}$.

Proof: This can be seen from the following which is obtained by performing a canonical transformation on $\underline{H}$

$$\underline{D}^{-1}\underline{H}\,\underline{D} = \begin{pmatrix} \underline{\Pi}(\underline{A}) - \underline{M}^{-1}\underline{B}\,\underline{R}^{-1}\underline{B}^{T}\underline{P}\,\underline{M} & -\underline{E} \\ 0 & -\underline{\Pi}(\underline{A}) - (\underline{M}^{-1}\underline{B}\,\underline{R}^{-1}\underline{B}^{T}\underline{P}\,\underline{M})^{T} \end{pmatrix}$$

where

$$\underline{E} = \underline{M}^{-1}\underline{B}\,\underline{R}^{-1}\underline{B}^{T}\underline{M}^{-T} \quad , \quad \underline{D} = \begin{pmatrix} \underline{M} & 0 \\ \underline{P}\,\underline{M} & \underline{M}^{-T} \end{pmatrix}, \tag{8}$$

$\underline{M}$ is the Modal matrix of $\underline{A}$, and $\underline{P}$ is the solution of AMRE.■

The problem of interest is to find a desired weighting matrix $\underline{Q}_d$, and an optimal control $\underline{U}$, for a given fixed $\underline{R}$, such that (2) is minimized and a set of desired closed loop eigenspectrum $\Lambda(\underline{A}_{cd})$ is achieved.

3.0 - REGULATOR WITH PREASSIGNED DEGREE OF STABILITY

Theorem 2- Given that the triplet $(\underline{A},\underline{B},\sqrt{\underline{Q}})$ is a minimum realization, and $\epsilon>0$, then the control law

$$\underline{U} = -\underline{R}^{-1}\underline{B}^{T}\,\tilde{\underline{P}}\,\underline{X} \tag{9}$$

where $\tilde{\underline{P}}$ is the solution of[1]

$$\tilde{\underline{P}}\,(\underline{A} + \epsilon\underline{I}) + (\underline{A}^{T} + \epsilon\underline{I})\tilde{\underline{P}} - \tilde{\underline{P}}\,\underline{B}\,\underline{R}^{-1}\underline{B}^{T}\tilde{\underline{P}} + \tilde{\underline{Q}} = 0 \tag{10}$$

with

$$\tilde{\underline{Q}} = \underline{Q} - 2\epsilon\underline{P} \tag{11}$$

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[1] Throughout the paper $\underline{I}$ denotes an identity matrix of appropriate dimension.

and $\underline{P}^{-} \leq 0$ being the unstable solution (Willems (1971)) of the AMRE in (5), will result in

$$\Lambda(\underline{A} - \underline{B}\,\underline{R}^{-1}\underline{B}^{T}\underline{\tilde{P}}) = \Lambda(\underline{A}_c) - 2\epsilon \qquad (12)$$

Proof: It is easy to show by direct substitution in (10), that the unstable solution of (10) is the same as that of (5), that is $\underline{\tilde{P}}^{-} = \underline{P}^{-}$. The proof is then accomplished using the symplectic properties of the following Hamiltonian system (see Amin (1985), or Medanic et.al. (1986)).

$$\underline{\tilde{H}} = \begin{pmatrix} \underline{A} + \epsilon\underline{I} & -\underline{B}\,\underline{R}^{-1}\underline{B}^{T} \\ -\underline{\tilde{Q}} & -(\underline{A}^{T} + \epsilon\underline{I}) \end{pmatrix} \qquad (13)\blacksquare$$

Remark 1 By comparing (5) and (6) it is obvious that $\underline{\tilde{P}}$ in (10) can be obtained from the Hamiltonian matrix in (13)

Remark 2 For a value of $\underline{Q} = \underline{Q}_d = \underline{\tilde{Q}} + 2\epsilon\underline{\tilde{P}}$ it can be shown that the optimal control law obtained from (3), for the minimization problem posed in (1-2) will be the same as the one obtained from (9).

Remark 3 Note that by appropriate selection of the ϵ, and $\underline{Q}_d$ as suggested in remark 2, the real parts of the entire closed loop eigenspectrum can be shifted. In the next sections this idea is applied for placing a subset of the eigenvalues at different locations in a recursive fashion.

4.0 - PLACING A SUBSET OF POLES

THEOREM 3- Consider the following reduced order (aggregate) 2ℓ x 2ℓ Hamiltonian system ($\ell<n$)[2]

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[2] From here on variables with the symbol "^" refer to the aggregated system.

$$\hat{H} = \begin{pmatrix} \hat{A} & -\hat{B}\,R^{-1}\hat{B}^T \\ -\hat{Q} & -\hat{A}^T \end{pmatrix} \tag{14}$$

obtained such that $\Lambda(\hat{A}) \subset \Lambda(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_\ell, \lambda_{\ell+1}, \ldots, \lambda_n\}$. For a choice of $\hat{Q}$ in the above, suppose Q in (6-a) is selected as $Q = C^T \hat{Q}\, C$, where C is an ℓxn, full rank matrix given by

$$C = [\, I_\ell \mid 0 \,]\, M^{-1} \tag{15}$$

and M is the modal matrix of A with its first ℓ columns being the eigenvectors $\{v_1, v_2, \ldots, v_\ell\}$ corresponding to $\{\lambda_1, \lambda_2, \ldots, \lambda_\ell\}$. Then for the given choice of Q, the eigenvalues of H are those of $\hat{H}$ plus the $(n-\ell)$ eigenvalues $\Lambda(A) \cap \Lambda(\hat{A})$ and their corresponding mirror image about the imaginary axis.

Proof: For the matrix $\hat{A}$ in (14) to retain the ℓ eigenvalues of A, the following consistency relation must be satisfied

$$C\,A = C\,A\,C^+C \tag{16}$$

where

$$C^+ = C^T(C\,C^T)^{-1} \tag{17}$$

is Moore-Penroze pseudo inverse of C. In which case

$$\hat{A} = C\,A\,C^+ \tag{18-a}$$

$$\hat{B} = C\,B \tag{18-b}$$

This is so because the choice of C given in (15) will make the subspace spanned by the ℓ rows of C to be orthogonal to that spanned by the eigenvectors $\{v_{\ell+1}, \ldots, v_n\}$, and therefore $\hat{A}$ will retain $\{\lambda_1, \ldots, \lambda_\ell\}$. (Aoki (1968), and Jamshidi (1983)).

For the given Q consider the following similarity transformation on H

$$\bar{H} = \Gamma^{-1} H \Gamma = \begin{pmatrix} \Pi(A) & -E \\ -M^T Q M & -\Pi(A) \end{pmatrix}$$

$$\Gamma = \begin{pmatrix} M & 0 \\ 0 & M^{-T} \end{pmatrix} \tag{19}$$

eigenvalues of $\bar{H}$ are the same as H and are given by letting $\det(SI-\bar{H})=0$

$$\det(SI-\bar{H}) = \det\begin{pmatrix} S\,I - \Pi(A) & E \\ M^T Q M & S\,I + \Pi(A) \end{pmatrix}$$

but, $\det\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \det(X)\det(W - Z\,X^{-1}Y)$ if X is nonsingular. Thus

$$\det(SI-\bar{H}) = \det[\,S\,I - \Pi(A)\,]\det[\,S\,I + \Pi(A) - M^T Q M\,(S\,I - \Pi(A))^{-1}E\,] \tag{20}$$

for the choice of $Q = C^T \hat{Q}\, C$, and C given by (15), after partitioning equation (20) will become

$$\det(SI-\bar{H}) = \det\left(\begin{array}{c|c} SI-\Pi_1 & 0 \\ \hline 0 & SI-\Pi_2 \end{array}\right)\det\left[\left(\begin{array}{c|c} SI-\Pi_1 & 0 \\ \hline 0 & SI-\Pi_2 \end{array}\right) - \left(\begin{array}{c|c} \hat{Q}\,(SI-\Pi_1)^{-1} & 0 \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c|c} E_1 & E_2 \\ \hline E_2 & E_3 \end{array}\right)\right] \tag{21}$$

where $\Pi_1 = \text{diag}\,[\lambda_1, \lambda_2, \ldots, \lambda_\ell]$, and $\Pi_2 = \text{diag}\,[\lambda_{\ell+1}, \ldots, \lambda_n]$

$$\det(SI-\bar{H}) = \det\left(\begin{array}{c|c} (SI-\Pi_1)\left[(SI+\Pi_1) - \hat{Q}\,(SI-\Pi_1)^{-1}E_1\right] & 0 \\ \hline 0 & (SI-\Pi_2)(SI+\Pi_2) \end{array}\right)$$

$$= \det\left[(SI-\Pi_1)\left[(SI+\Pi_1) - \hat{Q}\,(SI-\Pi_1)^{-1}E_1\right]\right]\det\left[(SI-\Pi_2)(SI+\Pi_2)\right] \tag{22}$$

Using a similar process as above it can easily be shown that the

eigenvalues of $\hat{\underline{H}}$ are given by the roots of the first polynomial in equation (22), and thus

$$\det(\underline{S}\,\underline{I} - \overline{\underline{H}}) = \det(\underline{S}\,\underline{I} - \hat{\underline{H}}) \det\left[(\underline{S}\,\underline{I} - \underline{\Pi}_2)(\underline{S}\,\underline{I} + \underline{\Pi}_2) \right] \qquad \blacksquare$$

The above theorem gives an indication as to how a set of eigenvalues of the system described in (1) and (2) can be placed at a particular location without altering the remaining eigenvalues. In addition, theorem (1) provides direction as to selecting the appropriate weighting matrix for placing the real poles or the real part of the complex conjugate pairs.

5.0 - PLACING A REAL POLE

This is simply achieved by using the transformation given in theorem (3) and obtaining the 2x2 aggregate Hamiltonian matrix. Then the appropriate $\hat{Q}$ matrix (in this case a scalar) will be selected using theorem (1).

6.0 - PLACING A COMPLEX CONJUGATE PAIR

Assume that at some stage during the recursive procedure it is desired to place the pair $-\alpha \pm j\beta$ at a new location. In such a case if (18-a) is used the matrix $\hat{\underline{A}}$ in (14) will be given by

$$\hat{\underline{A}} = \begin{pmatrix} -\alpha+j\beta & 0 \\ 0 & -\alpha-j\beta \end{pmatrix}$$

in order to work with real matrices consider the transformation $\underline{L}$ given by

$$\underline{L} = \begin{pmatrix} 0.5 & +j0.5 \\ 0.5 & -j0.5 \end{pmatrix}$$

where $\hat{\underline{A}}$ is now given by the real matrix

$$\hat{\underline{A}} = \underline{L}^{-1}\underline{C}\,\underline{A}\,\underline{C}^{+}\underline{L} = \begin{pmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{pmatrix} \tag{23-a}$$

and accordingly

$$\hat{\underline{B}} = \underline{L}^{-1}\underline{C}\,\underline{B} \tag{23-b}$$

also $\underline{Q}$ in the Hamiltonian matrix $\underline{H}$ should be selected as

$$\underline{Q} = \underline{C}^{T}\underline{L}^{-T}\,\hat{\underline{Q}}\,\underline{L}^{-1}\underline{C} \tag{23-c}$$

In order to place the complex conjugate pair of $-\alpha \pm j\beta$ at a new location $-\sigma \pm j\gamma$, it is necessary to investigate the effect of the weighting matrix on the imaginary part of the pair, and this is done by studying the characteristic equation of $\hat{\underline{H}}$ as a function of the weighting matrix $\hat{\underline{Q}}$. Before that is done a more general result due to Kwakernaak (1976) is given.

Assume that in equation (20), $\underline{M}^{T}\underline{Q}\,\underline{M} = q\bar{\underline{Q}}$, where q is a scalar and $\bar{\underline{Q}}$ is a constant, then the characteristic equation (20) can be written as

$$\det(\underline{S}\underline{I}-\bar{\underline{H}}) = \det(\underline{S}\underline{I}-\underline{\Pi})\,\det(\underline{S}\underline{I}+\underline{\Pi})\,\det\left(\underline{I} - q\left(\underline{S}\underline{I}+\underline{\Pi}^{T}\right)^{-1}\bar{\underline{Q}}\left(\underline{S}\underline{I}-\underline{\Pi}\right)^{-1}\underline{E}\right)$$

It is known that the stable roots of the above equation are the roots of the closed loop optimal system. Suppose now that $q \to 0$, in that case it is clear that the roots of the optimal closed loop regulator will approach the stable roots of

$$\det(\underline{S}\,\underline{I} - \underline{\Pi})\,\det(\underline{S}\,\underline{I} + \underline{\Pi}) = 0$$

which means that the unstable roots of the open loop system will shift to their mirror image location about the imaginary axis. However if $q \to \infty$

,then some of the poles approach finite locations and the others approach infinity and asymptotically group into Butterworth Configurations of different radii and orders.

Now consider the aggregate Hamiltonian system $\hat{\underline{H}}$, where a pair of complex conjugate poles of the system and their mirror image are retained in it. The characteristic polynomial for the Hamiltonian system $\hat{\underline{H}}$, with $\hat{\underline{Q}} = q\underline{I}$ is given by

$$\det(\underline{SI}-\hat{\underline{H}}) = \det(\underline{SI}-\underline{\Pi}_{-1}) \det(\underline{SI}+\underline{\Pi}_{-1}) \det(\underline{I} - q (\underline{SI}+\underline{\Pi}_{-1}^{T})(\underline{SI}-\underline{\Pi}_{-1}) \underline{E}_1)$$

$$= S^4-\left[2(\alpha^2-\beta^2)+q(e_1+ e_3)\right]S^2+q(e_1+ e_3)(\alpha^2+\beta^2)+q^2(e_1e_3- e_2^2)+(\alpha^2+\beta^2)^2 \qquad (24)$$

where

$$\underline{E}_1 = \begin{pmatrix} e_1 & e_2 \\ e_2 & e_3 \end{pmatrix}$$

is as in equation (21). A sequential prccedure is used to place the pair $-\alpha\pm j\beta$ at a new location $-\sigma\pm j\gamma$. Also, note that once $-\alpha\pm j\beta$ is placed, then through the property of the Hamiltonian system its mirror image will be placed accordingly.

According to the previous argument when q=0, the stable closed loop poles remain at their locations while the unstable ones will shift to their corresponding mirror image location. The approach here is then to increase q and find an appropriate value for it, if it exists, so that the imaginary part placement (γ) is achieved.

For a given γ by equating the coefficients of the like powers of S in equation (24) and a polynomial with roots $(-\delta\pm j\gamma,\delta\pm j\gamma)$, the following two equations in two unknown δ and q can be obtained (see also Solheim (1972))

$$q = \frac{2(\delta^2 + \beta^2 - \gamma^2 - \alpha^2)}{(e_1 + e_3)} \tag{25}$$

$$(\delta^2+\gamma^2)^2 = q(e_1+e_3)(\alpha^2+\beta^2) + q^2(e_1e_3-e_2^2) + (\alpha^2+\beta^2)^2 \tag{26}$$

Once q is solved for, the choice of $\hat{Q} = q\underline{I}$ will result in a set of eigenvalues $-\delta \pm j\gamma$ and their mirror image for the aggregate Hamiltonian system. At this time $\hat{Q}$ must be modified according to theorem (1) so that the placement of the real part (σ) is also accomplished.

In a similar manner it can be shown that if the pair $-\alpha \pm j\beta$ is to be placed at $-\nu$, and $-\mu$ locations, then given one of them, the value for appropriate q and the intermediate location of the other root φ can be obtained[3] . This is done as before by comparing the coefficients of the like powers of the characteristic polynomials as follows

$$q = \frac{(\varphi^2 + \nu^2) - 2(\alpha^2 - \beta^2)}{e_1 + e_3} \tag{27}$$

$$\varphi^2\nu^2 = q(e_1+e_3)(\alpha^2+\beta^2) + q^2(e_1e_3-e_2^2) + (\alpha^2+\beta^2)^2 \tag{28}$$

As before once q is solved for, the choice of $\hat{Q} = q\underline{I}$ will result in eigenvalues $-\nu$, and $-\varphi$ and their mirror image. Then the problem will reduce to placing a single real pole $-\varphi$ at $-\mu$.

The pole placement algorithm is complete by stating the following important lemma that will justify the sequential weight selection and pole placement procedure, and will also provide the missing link in between the recursive stages of the design process.

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3 If for some value of γ (ν), the value of δ (φ) become imaginary, then the pole placement for the given value of γ (ν) will not be possible.

Lemma 2- Assume $k+1 \leq n$ is the total number of stages in the sequential pole placement process. Given that

$$\underline{Q} = \underline{Q}_d = \sum_{i=0}^{k} \underline{Q}_{di} \tag{29}$$

where $\underline{Q}_{di}$'s are the desired weighting matrices that place a real or a complex conjugate pair at a desired location without altering the remaining poles at the stage i, and satisfy the following set of AMREs

$$\underline{P}_i\underline{A}_i + \underline{A}_i^T\underline{P}_i - \underline{P}_i\underline{B}\,\underline{R}^{-1}\underline{B}^T\underline{P}_i + \underline{Q}_i = 0 \tag{30}$$

where

$$\underline{A} = \underline{A}_0 \quad \text{and} \quad \underline{A}_{i+1} = \underline{A}_i - \underline{B}\,\underline{R}^{-1}\underline{B}^T\underline{P}_i = \underline{A}_i - \underline{B}\,\underline{K}_i \tag{31}$$

then the solution $\underline{P}$ of (5) can be obtained as

$$\underline{P} = \sum_{i=0}^{k} \underline{P}_i \tag{32}$$

and thus the optimal control gain

$$\underline{K}^* = \sum_{i=0}^{k} \underline{K}_i \tag{33}$$

Proof: The proof can be easily established by using (31) and adding up the equations in (30) to obtain

$$\sum_{i=0}^{k} (\underline{P}_i\underline{A} + \underline{A}^T\underline{P}_i - \sum_{j=0}^{k} \underline{P}_i\underline{B}\,\underline{R}^{-1}\underline{B}^T\underline{P}_j + \underline{Q}_i) = 0$$

Substituting (29), and (32) in the above and the fact that the solution to (5) is unique completes the proof.■

7.0 - POLE PLACEMENT ALGORITHM

The approach for pole placement is now summarized in the steps of the

following algorithm

1) Let $\underline{A}_0 = \underline{A}$, the sequential procedure starts at stage (i=0)

   a) If a real pole λ_o is to be placed at λ_d, use transformation given in (18) to obtain $\hat{\underline{A}}_i$, and $\hat{\underline{B}}_i$.

   b) If a complex conjugate pair $-\alpha \pm j\beta$ is to be assigned to $-\sigma \pm j\gamma$, use the transformation in (23-a,b) to obtain $\hat{\underline{A}}_i$, and $\hat{\underline{B}}_i$.

   c) If a complex conjugate pair $-\alpha \pm j\beta$ is to be assigned to two distinct real locations $-\nu$, and $-\mu$, use the transformation in (23-a,b) to obtain $\hat{\underline{A}}_i$ and $\hat{\underline{B}}_i$.

2) a) For a value of $\hat{Q}_i$, construct $\hat{\underline{H}}_i$ in (14), and obtain the unstable solution to the AMRE ($\hat{\underline{P}}^-$) described in theorem (1).

   b) Find the appropriate $\hat{\underline{Q}}_i - q\underline{I}$ to achieve the desired imaginary part γ using equations (25) and (26), also calculate and store δ. Using the value of $\hat{Q}_i$ obtained perform step 2a.

   c) Find the appropriate $\hat{\underline{Q}}_i - q\underline{I}$ to achieve one of the desired real poles (say $-\nu$) using (27) and (28), also calculate and store φ.

3) Calculate the value of ϵ in theorem (1)

   a) $\epsilon = (|\lambda_d| - |\lambda_c|)/\ 2$; where $\pm\lambda_c$ are the eigenvalues of $\hat{\underline{H}}_i$ with $\hat{\underline{Q}}_i$.

   b) $\epsilon = (|\sigma| - |\delta|)/\ 2$

   c) Go to step 5.

4) a,b) Calculate the appropriate $\hat{\underline{Q}}_i$ for proper pole placement according to (11), given here for convenience

$$\tilde{\underline{Q}}_i = \hat{\underline{Q}}_i - 2\epsilon\hat{\underline{P}} \tag{34}$$

5) a,b) Find the stable solution of the AMRE ($\tilde{\underline{P}}$) corresponding to the Hamiltonian system $\tilde{\underline{H}}_i$ given in (13).

c) Find the stable solution of the AMRE ($\hat{\underline{P}}$) corresponding to Hamiltonian system $\hat{\underline{H}}_i$ in (14).

6) Let $\hat{\underline{Q}}_{di}$ be the desired value of the weighting matrix that accomplishes the pole placement for the aggregated system, then as pointed out in remark 2, the value of this weighting matrix is

a,b) $$\hat{\underline{Q}}_{di} = \hat{\underline{Q}}_i + 2\epsilon\tilde{\underline{P}} \tag{35-a}$$

c) $$\hat{\underline{Q}}_{di} = \hat{\underline{Q}}_i \tag{35-b}$$

7) Calculate $\hat{\underline{K}}_i$ given by

a,b) $$\hat{\underline{K}}_i = \underline{R}^{-1}\hat{\underline{B}}_i^T\tilde{\underline{P}} \tag{36-a}$$

c) $$\hat{\underline{K}}_i = \underline{R}^{-1}\underline{B}_i^T\hat{\underline{P}} \tag{36-b}$$

8) Use the following to obtain the desired weighting matrix, and the optimal feedback gain for the original higher dimensional system

a)

$$\underline{Q}_{di} = \underline{C}^T\hat{\underline{Q}}_{di}\underline{C} \tag{37}$$

$$\underline{K}_i = \hat{\underline{K}}_i\underline{C} \tag{38}$$

b,c)

$$\underline{Q}_{di} = \underline{C}^T\underline{L}^{-T}\hat{\underline{Q}}_{di}\underline{L}^{-1}\underline{C} \tag{39}$$

$$\underline{K}_i = \hat{\underline{K}}_i\underline{L}^{-1}\underline{C} \tag{40}$$

9) Update the system's dynamics by using (31).

10) If all the eigenvalues are placed, stop and find $\underline{Q}_d$ from (29), and the

optimal gain from (33), otherwise let i=i+1, and go to step 1.

8.0 - EXAMPLES

Example 1- Consider the following fifth order dynamical system

$$\dot{\underline{X}} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 4 \\ -2 & -1 & 4 & 2 & -2 \\ 1 & 0 & 0 & 0 & 1 \\ 10 & -1 & 0 & 4 & -2 \end{pmatrix} \underline{X} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \underline{U}$$

This is an unstable system with open loop eigenspectrum of $\Lambda(\underline{A})$ = (-2.5405±j3.0409,3.6423±j0.7028,-.2037). Given that $\underline{R}$ is identity it is desired to find $\underline{Q}_d$ and the optimal control law such that the quadratic cost functional in (2) is minimized and at the same time $\Lambda(\underline{A}_{cd})$ = (-4±j2,-2±j0.5,-5) is achieved.

Solution

Lets start by placing the real pole λ_o = -0.2037, at λ_d= -5. To do this let $\underline{A} = \underline{A}_0$ and start the recursive procedure.

Stage i=0

The matrix $\underline{C}$ in (15) is obtained as

$$\underline{C} = [0.0226 \quad -0.2244 \quad 0.0534 \quad 0.8669 \quad -0.0765]$$

then using the transformation in (18)

$$\hat{\underline{A}}_0 = -0.2037$$

$$\hat{\underline{B}}_0 = [-0.0231 \quad 0.2778]$$

Note that as expected the mode λ_o= -0.2037 is retained in the reduced system. For a $\hat{\underline{Q}}_0$ = 1.0, the following aggregated Hamiltonian system is formed

$$\hat{H}_0 = \begin{pmatrix} -0.2037 & -0.0777 \\ -1.0 & 0.2037 \end{pmatrix}$$

from the above the unstable solution of the AMRE is obtained as

$$\hat{P}^- = -7.0653$$

also the closed loop poles of the aggregated system for $\hat{Q}_0 = 1.0$, is obtained as

$$\lambda_c = \pm 0.3452$$

Now the value of ϵ which places the pole at $\lambda_c = -0.3452$, to a new location $\lambda_d = -5.0$ is obtained as

$$\epsilon = (5.0-0.3452)/2 = 2.3274$$

the matrix $\hat{\tilde{Q}}_0$ in (34) is now given by

$$\hat{\tilde{Q}}_0 = 33.8877$$

the stable solution of the AMRE $\hat{\tilde{P}}_0$ is obtained from the Hamiltonian system in (13) given as

$$\hat{\tilde{H}}_0 = \begin{pmatrix} 2.1238 & -0.0777 \\ -33.8877 & -2.1238 \end{pmatrix}$$

$$\hat{\tilde{P}}_0 = 61.7442$$

the desired value of the weighting matrix $\hat{Q}_{d0}$ that accomplishes the pole placement for the reduced order system is obtained from (35-a) to be

$$\hat{Q}_{d0} = 321.2955$$

with the optimal feedback gain of

$$\hat{K}_0 = \begin{pmatrix} -1.4261 \\ 17.1497 \end{pmatrix}$$

for the original system therefore the weighting matrix, and the optimal gain are obtained from (37), and (38) to be

$$\underline{Q}_{d0} = \begin{pmatrix} 0.1641 & -1.6291 & 0.3875 & 6.2941 & -0.5552 \\ -1.6291 & 16.1757 & -3.8480 & -62.4954 & 5.5131 \\ 0.3875 & -3.8480 & 0.9154 & 14.8669 & -1.3115 \\ 6.2941 & -62.4954 & 14.8669 & 241.4529 & -21.3001 \\ -0.5552 & 5.5131 & -1.3115 & -21.3001 & 1.8790 \end{pmatrix}$$

$$\underline{K}_0 = \begin{pmatrix} -0.0322 & 0.3200 & -0.0761 & -1.2363 & 0.1091 \\ 0.3875 & -3.8480 & 0.9154 & 14.8669 & -1.3115 \end{pmatrix}$$

Note that the eigenvalues of the closed loop system $(\underline{A}_0 - \underline{B}\ \underline{K}_0)$ are given by $(-5.0, -2.5405 \pm j3.0409, 3.6423 \pm j0.7028)$, which as can be seen in the first iteration of the recursive procedure one of the poles is placed.

Stage i-1

$$\underline{A}_1 = \underline{A}_0 - \underline{B}\ \underline{K}_0 = \begin{pmatrix} 0 & 1.0 & 0 & 1.0 & 0 \\ 0.3875 & -3.8480 & 1.9154 & 14.8669 & 2.6885 \\ -2.3553 & 2.5280 & 3.1607 & -11.6306 & -0.7976 \\ 1.0 & 0 & 0 & 0 & 1.0 \\ 10.0322 & -1.3200 & 0.0761 & 5.2363 & -2.1091 \end{pmatrix}$$

At this stage it is desired to place the pair $3.6423 \pm j0.7028$, at $-2 \pm j0.5$. Using the transformation in (23), the following is obtained

$$\hat{\underline{A}}_1 = \begin{pmatrix} 3.6423 & -0.7028 \\ 0.7028 & 3.6423 \end{pmatrix}$$

$$\hat{\underline{B}}_1 = \begin{pmatrix} 0.9108 & 0.6028 \\ 0.6485 & -0.4903 \end{pmatrix}$$

now given $\alpha = 3.6423$, $\beta = 0.7028$, $\gamma = 0.5$, the appropriate value of q, for placing the imaginary part of the poles is calculated from (25), and (26). This leads to

$$q = 12.4484 \ ; \text{ and } \delta = 4.9559$$

therefore with $\hat{\underline{Q}}_1 = q\underline{I}$, the Hamiltonian system $\hat{\underline{H}}_1$ is formed, and after

calculating $\hat{P}^-$, and appropriate ϵ, the following is obtained from (34)

$$\hat{Q}_1 = \begin{pmatrix} 8.2925 & 0.2046 \\ 0.2046 & 8.0169 \end{pmatrix}$$

By forming the Hamiltonian matrix in (13) and obtaining the stabilizing solution of the AMRE, $\hat{P}_1$ the desired weighting matrix $\hat{Q}_{d1}$ is obtained to be

$$\hat{Q}_{d1} = \begin{pmatrix} -6.8983 & 4.5314 \\ 4.5314 & -18.8994 \end{pmatrix}$$

and the optimal feedback gain

$$\hat{K}_1 = \begin{pmatrix} 3.7313 & 4.5716 \\ 3.8155 & -5.3471 \end{pmatrix}$$

Finally using (39), and (40) we get

$$Q_{d1} = \begin{pmatrix} -284.4398 & -11.8650 & 35.0419 & -330.4378 & -66.0190 \\ -11.8650 & -1.1250 & -0.2890 & -12.3690 & -2.7680 \\ 35.0419 & -0.2890 & -9.1815 & 44.6398 & 8.0942 \\ -330.4378 & -12.3690 & 44.6398 & -387.0511 & -76.6636 \\ -66.0190 & -2.7680 & 8.0942 & -76.6636 & -15.3235 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 17.2291 & 2.3204 & 2.3281 & 16.4186 & 4.0347 \\ -21.0063 & -0.0568 & 4.8649 & -26.2435 & -4.8573 \end{pmatrix}$$

Stage i-2

$$A_2 = A_1 - B K_1 = \begin{pmatrix} 0 & 1.0 & 0 & 1.0 & 0 \\ -20.6187 & -3.9048 & 6.7803 & -11.3765 & -2.1688 \\ 1.4218 & 0.2643 & -4.0323 & -1.8058 & 0.0250 \\ 1.0 & 0 & 0 & 0 & 1.0 \\ -7.1969 & -3.6404 & -2.2520 & -11.1824 & -6.1438 \end{pmatrix}$$

In order to place the last pair ($-2.5405 \pm j3.0409$) at ($-4 \pm j2$), similar steps as in stage i-1 is taken and the desired weighting matrix and optimal gain are obtained as follow

$$Q_{d2} = \begin{pmatrix} -63.4466 & -23.0015 & -16.5283 & -75.9896 & -10.9232 \\ -23.0015 & -0.2704 & 8.3846 & -12.3882 & -1.0139 \\ -16.5283 & 8.3846 & 21.3113 & 7.2181 & 2.4039 \\ -75.9896 & -12.3882 & 7.2181 & -62.5252 & -7.5469 \\ -10.9232 & -1.0139 & 2.4039 & -7.5469 & -0.8048 \end{pmatrix}$$

$$K_2 = \begin{pmatrix} -5.6200 & -0.1392 & 1.9183 & -3.1643 & -0.2744 \\ -2.0259 & 0.6321 & 1.9073 & 0.1414 & 0.1542 \end{pmatrix}$$

At this point since all the poles are placed the recursive procedure is terminated and the resulting weighting matrix and optimal feedback gain is calculated from (29), and (33) as

$$Q_d = Q_0 + Q_1 + Q_2 = \begin{pmatrix} -347.7223 & -36.4957 & 18.9011 & -400.1332 & -77.4975 \\ -36.4957 & 14.7803 & 4.2476 & -87.2525 & 1.7312 \\ 18.9011 & 4.2476 & 13.0452 & 66.7248 & 9.1866 \\ -400.1332 & -87.2525 & 66.7248 & -208.1234 & -105.5106 \\ -77.4975 & 1.7312 & 9.1866 & -105.5106 & -14.2493 \end{pmatrix}$$

$$K^* = K_0 + K_1 + K_2 = \begin{pmatrix} 11.5769 & 2.5012 & 4.1703 & 12.0181 & 3.8693 \\ -22.6447 & -3.2726 & 7.6876 & -11.2351 & -6.0186 \end{pmatrix}$$

The control law $U^* = -K^* X$ with K^* given above will minimize the cost functional in (2) with $R = I$, and $Q = Q_d$, and will achieve the desired closed loop eigenspectrum $\Lambda_{cd} = (-5, -4\pm j2, -2\pm j0.5)$.

Example 2- The purpose of this example is to illustrate how a pair of complex conjugate poles can be placed at two distinct real positions. Consider the following MIMO system (Amin (1985)).

$$\dot{X} = \begin{pmatrix} -1.0 & 2.0 & 0.0 \\ 0.0 & -1.0 & 2.0 \\ -2.0 & -2.0 & -3.0 \end{pmatrix} X + \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix} U$$

The system has eigenspectrum $\Lambda(A) = (-3.0, -1\pm j2)$. It is desired to find an appropriate Q_d and an optimal control law so that the cost functional in

(2) with $\underline{R}$ being an identity matrix, is minimized. It is further desired to achieve the closed loop eigenspectrum $\Lambda_{cd} = \{-7,-4,-5\}$.

Solution

Stage i=0

Let us start by placing $\lambda_o = -3$, at $\lambda_d = -7$. This will be done by following the same procedure as in stage i=0, of the previous example. Doing so (with $\hat{Q}_0 = 1$) will result in

$$\underline{Q}_{d0} = \begin{pmatrix} 5.0 & 0.0 & 5.0 \\ 0.0 & 0.0 & 0.0 \\ 5.0 & 0.0 & 5.0 \end{pmatrix}$$

$$K_0 = \begin{pmatrix} 1.0 & 0.0 & 1.0 \\ 1.0 & 0.0 & 1.0 \end{pmatrix}$$

Stage i=1

$$\underline{A}_1 = \underline{A}_0 - \underline{B}\,\underline{K}_0 = \begin{pmatrix} -3.0 & 2.0 & -2.0 \\ 0.0 & -1.0 & 2.0 \\ -4.0 & -2.0 & -5.0 \end{pmatrix}$$

using the transformation in (23) the reduced order system is obtained as

$$\hat{\underline{A}}_1 = \begin{pmatrix} -1.0 & 2.0 \\ -2.0 & -1.0 \end{pmatrix}$$

$$\hat{\underline{B}}_1 = \begin{pmatrix} 0.20 & 0.20 \\ 1.40 & -0.60 \end{pmatrix}$$

for $\alpha=-1$, $\beta=2$, and $\nu=-4.0$, equations (27) and (28) are solved for q, and φ, and the result is

$$q = 149.21 \; ; \; \varphi = -18.3331$$

Then using $\hat{\underline{Q}}_1 = q\underline{I}$, the stabilizing solution of the AMRE ($\hat{\underline{P}}$) is obtained from the Hamiltonian system in (14), and the feedback gain $\hat{\underline{K}}_1$ is found, then $\hat{\underline{Q}}_{d1}$ and $\hat{\underline{K}}_1$ are transformed to higher dimensional system using (39), and (40), in which case

$$\underline{Q}_{d1} = \begin{pmatrix} 59.6840 & 29.8420 & -29.8420 \\ 29.8420 & 149.2100 & 29.8420 \\ -29.8420 & 29.8420 & 29.8420 \end{pmatrix}$$

and

$$\underline{K}_1 = \begin{pmatrix} 6.2929 & 10.7492 & -0.6122 \\ 2.7381 & -4.4859 & -3.3207 \end{pmatrix}$$

finally the pole at $\varphi = -18.3331$, is to be placed at $\mu = -5$. The problem of placing the complex conjugate pair now has been reduced to placing a single real pole, for which the solution already has been discussed.

Stage i=2

$$\underline{A}_2 = \underline{A}_1 - \underline{B}\,\underline{K}_1 = \begin{pmatrix} -12.031 & -4.2633 & 1.9329 \\ -3.5548 & -16.235 & -0.7085 \\ -13.031 & -8.2633 & -1.0671 \end{pmatrix}$$

Following the procedure as before (and using $\hat{Q}_2 = 1$), the following is obtained

$$\underline{Q}_{d2} = \begin{pmatrix} -34.3041 & -65.1010 & 1.1691 \\ -65.1010 & -123.5461 & 2.2186 \\ 1.1691 & 2.2186 & -0.0398 \end{pmatrix}$$

$$\underline{K}_2 = \begin{pmatrix} -4.2101 & -7.9899 & 0.1435 \\ 1.3700 & 2.5999 & -0.0467 \end{pmatrix}$$

At this stage the recursive procedure is terminated, and the desired weighting matrix and the optimal feedback gain is calculated as

$$\underline{Q}_d = \begin{pmatrix} 30.3799 & -35.2590 & -23.6729 \\ -35.2590 & 25.6639 & 32.0606 \\ -23.6729 & 32.0606 & 34.8022 \end{pmatrix}$$

$$\underline{K}^* = \begin{pmatrix} 3.0828 & 2.7593 & 0.5313 \\ 5.1081 & -1.8860 & -2.3674 \end{pmatrix}$$

Remark 4 It is interesting to note that the above weighting matrix is sign indefinite. It turns out that by using the weight selection strategy presented in this study, it is possible in some cases to arrive at an

indefinite, or even negative definite Q matrices. This class of sign-indefinite Q's do not pose any threat of instability, since the weighting matrix Q is preselected for desired stable locations of the closed loop eigenspectrum. Although we have not attempted to explore this class weighting matrices in more depth yet, the reader can refer to a recent work by Johnson (1987, 1988) where the hidden regions of "unreachable poles" when using standard LQR theory are discussed. These are the regions that can not be realized as optimum closed loop poles using the customary positive (semi) definite Q. It is pointed out however, that this defect may be removed by using a sign-indefinite or a negative definite Q. Therefore, the sign-indefinite weights that are may result using the proposed method fall under the same class of weights discussed by Johnson (1987,1988). In summary therefore, it is concluded that a positive definite or positive semi definite requirement on the weighting matrix Q is not a necessary constraint in the proposed approach.

9.0 - CONCLUSION

The freedom in choosing the weighting matrices of an optimal linear quadratic regulator has been utilized during a recursive design procedure. The proper weighting matrix on the states, and an optimal controller are found such that the control law would minimize the quadratic performance measure, and at the same time would result in a set of desired closed loop eigenspectrum. The attractive features of this approach are that it uses model reduction, so that almost the entire computations involve only second or fourth order matrices depending on whether a real pole or a complex conjugate pair is being shifted. It also uses the symplectic properties of the Hamiltonian system for pole shifting and computaticn of the stable and unstable solutions of the matrix Riccati equation. In addition it was shown that frequencies of oscillation ($<\beta$) can be

achieved, and thus it is possible to place the imaginary part as well as the real part of an open loop pair. Also it is possible to convert a complex conjugate pair into two real distinct poles. Finally, it was shown that the customary requirement of having a positive definite (or semidefinite) Q matrix is not a necessary one.

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